

Massive Nambu-Goldstone Bosons

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Nicolis and Piazza have recently pointed out the existence of Nambu-Goldstone-like excitations in relativistic systems at finite density, whose gap is *exactly* determined by the chemical potential and the symmetry algebra. We show that the phenomenon is much more general than anticipated and demonstrate the presence of such modes in a number of systems from (anti)ferromagnets in magnetic field to superfluid phases of quantum chromodynamics. Furthermore, we prove a counting rule for these massive Nambu-Goldstone bosons and construct a low-energy effective Lagrangian that captures their dynamics.

Dedicated to Jiří Hošek on the occasion of his 70th birthday.

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Introduction.—Trying to understand collective behavior of matter in nonlinear many-body systems is a challenge common to many areas of physics. At long distances and low temperatures, excitations with vanishing or small gap (mass) dominate the dynamics. The concept of spontaneous symmetry breaking has been crucial for its understanding, as it unambiguously predicts existence of gapless excitations—the Nambu-Goldstone bosons (NGBs)—such as phonons or magnons. For nearly five decades, however, their correct counting and dispersion relations eluded consistent understanding. Recently, the present authors developed a unified framework to determine the number and dispersion relations of NGBs [1, 2], including their redundancies [3].

Cases where exact statements can be made about gapped modes are rare though. Kohn’s theorem states that a gas of charged particles with Galilean invariance, when exposed to a uniform magnetic field, sustains a collective mode with the cyclotron gap [4]. Moreover, some soliton solutions to nonlinear equations saturate Bogomol’nyi-Prasad-Sommerfield bounds, allowing their energies to be determined based on symmetry alone [5], albeit with limited applicability to observable systems. NGBs perturbed by explicit symmetry breaking effects acquire small gaps and are usually called pseudo-NGBs [6]. Yet their gaps in general can be computed only approximately.

Recently, Nicolis and Piazza [7] pointed out that the gaps of pseudo-NGBs can be determined in special circumstances. Considering Lorentz-invariant systems perturbed only by a chemical potential whose charge operator is spontaneously broken, they showed that the mass of the associated pseudo-NGB can be computed exactly and is free of radiative corrections. We will call such state *massive NGB* (mNGB). In the present Letter, we show that mNGBs appear in a much broader class of systems; the theory need not be Lorentz-invariant, or the chemical potential operator spontaneously broken. We provide a counting rule for the number of mNGBs and construct an effective Lagrangian description for them.

General argument.—Consider a many-body system specified by the Hamiltonian H with an internal symmetry group G . In order to describe states with finite charge density, it is customary to introduce a chemical potential μ by $\tilde{H} \equiv H - \mu Q$, where Q is one of the generators of G . The vacuum $|0\rangle$ is defined as the eigenstate of \tilde{H} with the lowest eigenvalue. Without lack of generality, we can take $\tilde{H}|0\rangle = 0$. Since the generators Q_i of the Lie group G commute with the Hamiltonian H , they are all time-independent in the Heisenberg picture defined by H [8],

$$Q_i(t) \equiv \int d\mathbf{x} e^{iHt-i\mathbf{P}\cdot\mathbf{x}} j_i^0(0) e^{-iHt+i\mathbf{P}\cdot\mathbf{x}}, \quad (1)$$

where $j_i^0(x)$ are the corresponding local charge densities.

When spontaneously broken, generators of the symmetry group \tilde{G} of the full Hamiltonian \tilde{H} give rise to standard massless NGBs. On the other hand, spontaneously broken generators that do not commute with Q create pseudo-NGBs whose gaps are proportional to μ . The novelty of the observation made by Nicolis and Piazza [7] is that the proportionality constants can be computed *exactly* by group theory.

By the standard Cartan decomposition, explicitly broken generators can be split into pairs $Q_{\pm\sigma}$ —the roots—such that

$$[Q, Q_{\pm\sigma}] = \pm q_\sigma Q_{\pm\sigma}, \quad (2)$$

where $Q_{\pm\sigma}$ are some complex linear combinations of explicitly broken generators and $(Q_{\pm\sigma})^\dagger = Q_{\mp\sigma}$. Let us now focus on the quantity $\lambda_\sigma \equiv \langle 0|[Q_{+\sigma}(t), j_{-\sigma}^0(0)]|0\rangle$, which is manifestly time-independent. Using Eq. (1), inserting a complete set of eigenstates $|n, \mathbf{p}\rangle$ of momentum \mathbf{P} and energy \tilde{H} , and carrying out integration over space, we obtain

$$\begin{aligned} \lambda_\sigma &= \sum_n e^{-i[E_n(\mathbf{0})-\mu q_\sigma]t} |\langle 0|j_{+\sigma}^0(0)|n, \mathbf{0}\rangle|^2 \\ &\quad - \sum_n e^{-i[E_n(\mathbf{0})+\mu q_\sigma]t} |\langle 0|j_{-\sigma}^0(0)|n, \mathbf{0}\rangle|^2. \end{aligned} \quad (3)$$

Provided that $\mu_{q\sigma} > 0$, time-independence of the left-hand side and $E_n(\mathbf{0}) \geq 0$ require $\langle 0|j_{-\sigma}^0(0)|n, \mathbf{0}\rangle = 0$ for each n . If λ_σ is zero, implying $\langle 0|j_{+\sigma}^0(0)|n, \mathbf{0}\rangle = 0$ for each n as well, $Q_{\sigma R} \equiv (Q_{+\sigma} + Q_{-\sigma})/\sqrt{2}$ and $Q_{\sigma I} \equiv (Q_{+\sigma} - Q_{-\sigma})/\sqrt{2}i$ cannot be spontaneously broken. Namely, there is no local field $\Phi(x)$ such that $\langle 0|[Q_{\sigma R, I}(t), \Phi(0)]|0\rangle \neq 0$.

On the other hand, if $\lambda_\sigma \neq 0$, $Q_{\sigma R}$ and $Q_{\sigma I}$ are broken spontaneously and there must be a state $|n, \mathbf{0}\rangle$ with mass

$$\tilde{H}|n, \mathbf{0}\rangle = E_n(\mathbf{0})|n, \mathbf{0}\rangle = \mu_{q\sigma}|n, \mathbf{0}\rangle \quad (4)$$

such that $\langle 0|j_{+\sigma}^0(0)|n, \mathbf{0}\rangle \neq 0$ and $\langle 0|j_{-\sigma}^0(0)|n, \mathbf{0}\rangle = 0$. This is the mNGB associated with the pair $Q_{\pm\sigma}$.

Our derivation clarifies several points on mNGBs. First, it is clear that the assumption of the underlying dynamics being Lorentz-invariant [7] can be dropped. Also, λ_σ always plays the role of the order parameter for charges $Q_{\sigma R, I}$. Finally, note that $\langle 0|j_{-\sigma}^0(0)|n, \mathbf{0}\rangle = 0$ for all n means $Q_{+\sigma}|0\rangle = 0$, while if λ_σ is nonzero, $Q_{-\sigma}|0\rangle \neq 0$. This observation leads to a simpler, albeit less rigorous, understanding of mNGBs. Eq. (2) gives $[\tilde{H}, Q_{\pm\sigma}] = \mp\mu_{q\sigma}Q_{\pm\sigma}$ which implies that the state $Q_{-\sigma}|0\rangle$ has energy $\mu_{q\sigma}$. As there cannot be a state with energy lower than the vacuum, $Q_{+\sigma}|0\rangle$ has to vanish. Our argument is reminiscent of the proof of Kohn's theorem [4], allowing for a unified comprehension of the two phenomena.

Number of mNGBs.—For a proper understanding of the low-energy dynamics of the system, it is important to know the number and dispersion relations of NGBs. Denoting the broken generators of \tilde{G} as \tilde{Q}_a , the former is given by [1, 2],

$$n_{\text{NGB}} = n_A + n_B, \quad n_A = n_{\text{BG}} - \text{rank } \tilde{\rho}, \quad n_B = \frac{1}{2}\text{rank } \tilde{\rho}, \quad (5)$$

where n_{BG} is the number of broken generators and

$$\tilde{\rho}_{ab} \equiv -i \lim_{\Omega \rightarrow \infty} \frac{1}{\Omega} \langle 0|[\tilde{Q}_a, \tilde{Q}_b]|0\rangle, \quad (6)$$

Ω being the spatial volume. The type-A and B NGBs generically have linear and quadratic dispersion relations and correspond to type-I and II in the Nielsen-Chadha theorem [9], even though this is not always the case [10]. Each type-B NGB is described by a canonically conjugate pair of broken generators \tilde{Q}_a and \tilde{Q}_b with nonzero $\tilde{\rho}_{ab}$ and hence two broken symmetries count as one degree of freedom, whereas type-A NGBs are stand-alone like in the original Goldstone theorem.

Here we address the question of counting the mNGBs [11]. Namely, we show that their number is given by

$$n_{\text{mNGB}} = \frac{1}{2}(\text{rank } \rho - \text{rank } \tilde{\rho}), \quad (7)$$

where the matrix ρ is defined analogously to Eq. (6) for all generators of G instead of just \tilde{G} . To that end, we have to further specify the structure of the Lie algebra. First, let us choose the maximal number of mutually commuting generators of \tilde{G} , including Q itself, to form the Cartan subalgebra. By a proper choice of the vacuum $|0\rangle$, we can always ensure that these are the only generators

of \tilde{G} that can possibly have a nonzero vacuum expectation value [1]. This alone does not preclude the possibility that explicitly broken generators acquire expectation values. However, since $\pm\mu_{q\sigma}\langle 0|Q_{\pm\sigma}|0\rangle = \langle 0|[\mu Q, Q_{\pm\sigma}]|0\rangle = \langle 0|[\tilde{H}, Q_{\pm\sigma}]|0\rangle = 0$ thanks to $\tilde{H}|0\rangle = 0$, $\langle 0|Q_{\pm\sigma}|0\rangle$ must vanish for any nonzero q_σ . If we arrange the generators as $Q_i = (Q_{1R}, Q_{1I}, \dots, Q_{mR}, Q_{mI}, \tilde{Q}_1, \dots, \tilde{Q}_{\dim \tilde{G}})$, where $m \equiv (\dim G - \dim \tilde{G})/2$, the matrix ρ becomes block-diagonal, $\rho = \text{diag}(i\tau_2\lambda_1, \dots, i\tau_2\lambda_m, \tilde{\rho})$, τ_2 being the second Pauli matrix. Thus, $\frac{1}{2}(\text{rank } \rho - \text{rank } \tilde{\rho})$ counts the number of nonzero λ_σ 's. Assuming that there is at most one mNGB for each pair of $Q_{\pm\sigma}$, this proves our counting rule (7). In the following, we provide examples of mNGBs, demonstrating the validity of Eq. (7) in physically interesting systems [12].

Ferromagnet.—The Hamiltonian of a ferromagnet enjoys the internal $G = \text{O}(3)$ symmetry group of spin rotations. In the ground state, individual spins are aligned, breaking this symmetry down to its $\text{O}(2)$ subgroup. The two broken generators give rise to a single type-B NGB with a quadratic dispersion relation at low momentum—the magnon [2, 9].

Consider now switching on a uniform magnetic field \mathbf{B} oriented in the z -direction. This amounts to breaking the symmetry explicitly to $\tilde{G} = \text{O}(2)$ by adding to the Hamiltonian $-\mu_m B S_z$ ($\mu_m B > 0$), where \mathbf{S} is the total spin operator and μ_m is the magnetic moment. This term can be viewed as a chemical potential $\mu = \mu_m B$ for the generator $Q = S_z$. Given that $[S_z, S_\pm] = \pm S_\pm$ where $S_\pm \equiv (S_x \pm iS_y)/\sqrt{2}$, S_- must excite a mNGB of gap μ , which is just the magnon with energy lifted by the magnetic field [13]. The operator S_+ annihilates the ground state. Both these assertions are easy to understand from the fact that the vacuum corresponds to the state with maximum spin in the direction of the magnetic field, and the magnon to an excitation caused by flipping one of the spins. Note that the counting rule (7) predicts the correct number of mNGBs, that is, $n_{\text{mNGB}} = \frac{2-0}{2} = 1$.

Antiferromagnet.—In the absence of a magnetic field, assume the spins are oriented alternately along the z -axis; $G = \text{O}(3)$ is broken to $\text{O}(2)$ just like in a ferromagnet. In this case there are two type-A NGBs, one for each broken generator.

Applying a magnetic field along the z -axis leads to an instability as the NGBs attempt to acquire masses $\pm\mu = \pm\mu_m B$. The ground state rearranges with alternating spins pointing in an orthogonal direction instead, say along the x -axis. Then $Q = S_z$ is a spontaneously broken generator which commutes with \tilde{H} and creates a gapless type-A NGB. On the other hand, the other spontaneously broken generator S_y is explicitly broken at the same time, creating a mNGB with gap μ [14]. The magnetic field induces a small magnetization along the z -axis, and hence $\rho_{xy} = \langle 0|[S_x, S_y]|0\rangle \neq 0$. Consequently, $n_{\text{mNGB}} = \frac{2-0}{2} = 1$, consistent with Eq. (7).

Relativistic Bose-Einstein condensation.—As an explicit example where $\tilde{\rho} \neq 0$, consider a theory of a complex scalar doublet ϕ with a global $\tilde{G} = \text{SU}(2) \times \text{U}(1)$ symmetry,

$$\mathcal{L} = D_\mu \phi^\dagger D^\mu \phi - M^2 \phi^\dagger \phi - \lambda(\phi^\dagger \phi)^2, \quad (8)$$

where $D_0 \phi \equiv (\partial_0 - i\mu)\phi$. This model features a relativis-

tic Bose-Einstein condensation (BEC) phase for $\mu > M$, in which the symmetry is spontaneously broken to a $U(1)'$ subgroup. The three broken generators produce two NGBs, one type-A and one type-B [15], consistent with Eq. (5), since one of the $SU(2)$ charges develops nonzero density in the ground state, hence $\text{rank } \tilde{\rho} = 2$.

The type-B NGB in this model has an “antiparticle”, carrying opposite charge. Its mass equals 2μ and does not receive radiative corrections [16]. To see why, note that when $\mu = 0$, the Lagrangian enjoys an extended internal symmetry, $G = SO(4) \simeq SU(2)_L \times SU(2)_R$. This is most easily seen by defining a 2×2 matrix $\Phi = (\phi, i\tau_2 \phi^*)$, which transforms under G as $\Phi \rightarrow U_L \Phi U_R^\dagger$. Denote the generators of G as \vec{L} and \vec{R} , respectively; they are both given by a half of the Pauli matrices. The $SU(2)$ rotations of the doublet ϕ now correspond to $SU(2)_L$; the $U(1)$ phase transformations are generated by $2R_3$. The remaining two generators of $SU(2)_R$ are explicitly broken by the chemical potential μ . In the BEC phase, the condensate can be chosen as $\langle 0|\Phi|0 \rangle \sim \mathbb{1}$ so that they are also broken spontaneously. Since the R_\pm generators of $SU(2)_R$ satisfy the commutation relation $[2R_3, R_\pm] = \pm 2R_\pm$, Eq. (3) implies that R_- creates a mNGB with mass 2μ , in agreement with the explicit calculation. Indeed, $m_{\text{mNGB}} = \frac{4-\mu}{2} = 1$. This example obviously admits a generalization to a large class of relativistic linear sigma models with chemical potential [17, 18], the key ingredient being an extended global symmetry when the chemical potential is set to zero.

QCD-like theories.—Quantum ChromoDynamics (QCD) with two degenerate quark flavors possesses an approximate global $SU(2)_L \times SU(2)_R$ chiral symmetry. A nonzero quark mass breaks this explicitly to the $G = SU(2)_V$ subgroup generated by $\vec{V} \equiv \vec{R} + \vec{L}$. The chiral condensate in the QCD vacuum breaks the symmetry spontaneously in the same way, resulting in three pseudo-NGBs in the spectrum: the pions.

Nonzero chemical potential, μ_1 , for V_3 breaks the exact symmetry G further to its $\tilde{G} = U(1)_I$ subgroup, generated by V_3 . While the mass of the neutral pion is insensitive to μ_1 , the masses of the charged pions become $m_\pi \pm \mu_1$. Consequently, once $\mu_1 > m_\pi$, the positively charged pion undergoes BEC, breaking the residual \tilde{G} symmetry spontaneously [19]. Therefore, the spectrum in the pion BEC phase exhibits one true, type-A NGB. However, the ground state has a nonzero isospin density, $\langle 0|V_3|0 \rangle = -i\langle 0|[V_1, V_2]|0 \rangle$, and Eq. (7) implies that there is also one mNGB. Such a state has indeed been found using effective field theory [20] as well as various model approaches [18, 21] and can be identified with the neutral pion in the superfluid medium. As opposed to these approximate calculations, the result of Ref. [7] nevertheless ensures that its mass is exactly equal to μ_1 . This follows from the commutation relation $[V_3, V_\pm] = \pm V_\pm$.

In the limit of massless quarks, the full symmetry becomes $G = SU(2)_L \times SU(2)_R$; isospin chemical potential breaks this explicitly to $\tilde{G} = U(1)_L \times U(1)_R$. Pion condensate now develops at any nonzero chemical potential, breaking \tilde{G} spontaneously to $U(1)$. Thus, there is one type-A

NG boson in the spectrum. Moreover, given the commutators $[V_3, R_\pm] = \pm R_\pm$ and $[V_3, L_\pm] = \pm L_\pm$, we find that $\langle 0|V_3|0 \rangle = -2i\langle 0|[R_1, R_2]|0 \rangle = -2i\langle 0|[L_1, L_2]|0 \rangle \neq 0$, as a result of which there are $\frac{4-\mu}{2} = 2$ mNGBs according to Eq. (7). This is consistent with explicit calculations; the additional mNGB has the quantum numbers of the σ meson.

A related theory that has attracted attention in connection to lattice Monte-Carlo simulations at nonzero density is two-color QCD. Thanks to extended global symmetry, its two-flavor version has altogether five pseudo-NGBs in the vacuum: three pions and an (anti)diquark [22]. For baryon chemical potentials μ_B exceeding m_π , the diquark will thus condense. Using the same argument as above, one can show that in this BEC phase, the pions form a triplet of mNGBs with mass exactly equal to μ_B . Such modes have been observed in analytic calculations [22, 23] as well as on the lattice [24]. In the limit of massless quarks, the enlarged symmetry again leads to the presence of an extra mNGB with the quantum numbers of the σ meson. Similar conclusions can be reached for an arbitrary even number of flavors [22].

Effective Lagrangian formalism.—For energies, including the chemical potential μ , well below the characteristic scale of spontaneous symmetry breaking, the dynamics of both NGBs and mNGBs can be described using effective field theory [13, 25]. Assuming spatial rotational invariance and using the fact that the chemical potential can be regarded as a constant temporal gauge field [26], $A_0^i = \mu^i$ ($i = 1, \dots, \dim G$), the leading-order effective Lagrangian reads [13]

$$\begin{aligned} \mathcal{L}_{\text{eff}} = & c_a(\pi) \dot{\pi}^a + e_i(\pi) \mu^i + \frac{1}{2} \bar{g}_{ab}(\pi) D_t \pi^a D_t \pi^b \\ & - \frac{1}{2} g_{ab}(\pi) \nabla \pi^a \cdot \nabla \pi^b. \end{aligned} \quad (9)$$

The NG fields π^a ($a = 1, \dots, \dim G/H$) live on the coset G/H of the broken symmetry, H being the unbroken subgroup at zero chemical potential; $g_{ab}(\pi)$ and $\bar{g}_{ab}(\pi)$ are both G -invariant metrics on the coset. Under an infinitesimal symmetry transformation defined by a set of parameters ϵ^i , the coset fields change as $\delta \pi^a = \epsilon^i h_i^a(\pi)$, where $h_i^a(\pi)$ are the Killing vectors of the metrics. The covariant derivative is $D_t \pi^a \equiv \dot{\pi}^a - \mu^i h_i^a(\pi)$.

Explicit expressions can be obtained using the formalism developed in Ref. [27]. Denoting now the broken group generators as T_a and the unbroken generators as T_ρ , we represent the coset element by $U(\pi) \equiv e^{iT_a \pi^a}$ and define the Maurer-Cartan form as $\omega_a(\pi) = T_i \omega_a^i(\pi) \equiv -iU(\pi)^\dagger \frac{\partial}{\partial \pi^a} U(\pi)$. Assuming that G is compact and that the generators are normalized according to $\text{Tr}(T_i T_j) = \lambda \delta_{ij}$, we obtain [28],

$$\begin{aligned} g_{ab}(\pi) &= g_{cd}(0) \omega_a^c(\pi) \omega_b^d(\pi), \quad e_i(\pi) = \nu_i^j(\pi) e_j(0), \\ h_i^a(\pi) \omega_a^b(\pi) &= \nu_i^b(\pi), \quad c_a(\pi) = -\omega_a^i(\pi) e_i(0), \end{aligned} \quad (10)$$

where $\nu_i^j(\pi) \equiv \frac{1}{\lambda} \text{Tr}[U(\pi)^\dagger T_i U(\pi) T_j]$. For consistency with the G -invariance of the action, the effective couplings $e_i(0)$ and $g_{ab}(0)$ must satisfy $f_{i\rho}^j e_j(0) = 0$ and $f_{\rho a}^c g_{cb}(0) +$

$f_{pb}^c g_{ac}(0) = 0$, where f_{ij}^k are the structure constants of the symmetry group. Similar expressions hold for \bar{g} .

Let us emphasize that while the effective Lagrangian (9) was derived by Leutwyler nearly two decades ago [13], he only obtained a set of differential equations determining the effective couplings $c_a(\pi)$, $e_i(\pi)$, $g_{ab}(\pi)$, $\bar{g}_{ab}(\pi)$ implicitly. To the best of our knowledge, this is the first time that a general solution has been found, maintaining the full nonlinear dependence of the couplings on the coset fields π^a . Using Eq. (10), the effective Lagrangian is now completely fixed by the values of $g_{ab}(0)$ and $\bar{g}_{ab}(0)$, encoding decay constants of the NGBs, and of $e_i(0)$, expressing charge densities in the ground state.

The effect of the chemical potential on the ground state and spectrum can be completely deduced from these expressions. First of all, the Lagrangian (9) acquires a potential

$$V(\pi) = -v^i(\pi)e_i(0) - \frac{1}{2}\bar{g}_{ab}(0)v^a(\pi)v^b(\pi), \quad (11)$$

where $v^i(\pi) \equiv \mu^j \nu_j^i(\pi)$. As observed above, turning on the chemical potential can lead to an instability of the vacuum. The true ground state is then found by minimizing the potential. In particular, when $e_i(0) = 0$ for all i , chemical potentials for unbroken generators T_ρ generate the potential $-\frac{1}{2}\bar{g}_{cd}(0)(\mu^\rho f_{a\rho}^c \pi^a)(\mu^\sigma f_{b\sigma}^d \pi^b) + \mathcal{O}(\pi^4)$, which is negative definite so that NGBs condense. This is what happens when an antiferromagnet is subjected to a magnetic field along the z -axis: BEC of NGBs leads to a (canted) Néel order in the xy plane. On the other hand, when $e_i(0) \neq 0$ for some i , BEC of NGBs always results in a new ground state where μ^i is parallel to $e_i(0)$, at least for an infinitesimal μ^i . For example, in the case of ferromagnet where $e_z(0) = \langle 0|S_z|0\rangle/\Omega > 0$ dominates, a magnetic field in x -direction causes rearrangement of the magnetization into x -direction, while a magnetic field in z -direction further stabilizes the ground state.

The case of ferromagnet admits a straightforward generalization. Consider a system which features: (i) an unbroken $U(1)$ symmetry; (ii) a type-B NGB charged under it; (iii) no type-A NGBs charged under it. Then, introducing a chemical potential for the unbroken subgroup with a suitably chosen sign will lift the energy of the type-B NGB, making it a mNGB. No instability occurs since a type-B NGB is described by a pair of canonically conjugate NG fields; there is no gapless mode charged oppositely under the unbroken $U(1)$.

Provided that $e_i(0) = 0$ for all i , simple results can be obtained also in the case that chemical potentials are introduced for the broken generators T_a . Assuming that the coset G/H is a symmetric space and NGBs transform irreducibly under the unbroken subgroup H , the first-order derivative term starts from $\mathcal{O}(\pi^3)$. The potential term reads $\frac{1}{2}\mu^c \mu^d f_{ca}^\rho f_{db}^\rho \pi^a \pi^b$, explaining the mass of the mNGBs. Of course, the applicability of our effective Lagrangian is not restricted to this particular case; we checked that all examples we mentioned above can be explained by using our formalism.

Quasiparticle nature of mNGBs.—What happens to the mNGB when μ exceeds the symmetry-breaking scale? The general argument asserts just the existence of a state which, in

the long-wavelength limit, is an exact eigenstate of the Hamiltonian. True NGBs are actually well-defined quasiparticles even at finite wavelength, as the ratio of their width and their energy goes to zero in the zero-momentum limit [16].

First, note that in all the examples discussed above, the mNGB is stable at low enough chemical potential. The examples where Q is not spontaneously broken are trivial as the mNGB is then the lightest mode carrying its charge. Even in other cases, the mNGB is protected against decay by an unbroken symmetry. Indeed, the mNGB in the $SU(2) \times U(1)$ linear sigma model (8) is stable due to conservation of the unbroken $U(1)'$ charge. The pion superfluid phase of QCD with isospin chemical potential has an unbroken discrete symmetry: a combination of parity with a $U(1)_I$ rotation by the angle π ; the neutral pion is the only NGB mode odd under this transformation. Finally, in the diquark BEC phase of two-color QCD with baryon chemical potential, the pion triplet is stable as a consequence of isospin and parity conservation.

However, the mNGB is not expected to remain stable for arbitrarily high chemical potential. Consider the case of QCD with isospin chemical potential as an example. At very high μ_I , the system behaves as a weakly coupled superfluid. The quarks are gapped, but although their gap Δ increases with μ_I , the ratio Δ/μ_I tends to zero with increasing μ_I [19]. The mNGB will thus dissolve into a pair of quarks. This is in no contradiction with the general result, only its implications are trivial at high chemical potential: the mNGB excitation is realized as a state in a two- or many-body continuum. In general, the mNGB mode is expected to be a well-defined quasiparticle only for chemical potentials well below the characteristic scale of spontaneous symmetry breaking.

Conclusions.—We have generalized the argument of Ref. [7] and shown that a large class of quantum many-body systems features mNGBs whose mass is exactly determined by the chemical potential and the symmetry algebra. Since the chemical potential acts as a constant background gauge field, this can be understood as a consequence of gauge invariance, encoded in the effective Lagrangian (9). Moreover, we have derived a general expression for the number of mNGBs.

Of course, the calculation of the mass and the discussion of stability presented here is merely the first step towards a more detailed analysis of the properties of mNGBs. It would be certainly interesting to investigate which of the other hallmarks of NGBs are preserved in presence of the chemical potential explicitly breaking part of the symmetry. These include for instance the saturation of Ward-Takahashi identities for correlation functions of the broken currents and low-energy theorems for the scattering amplitudes involving one or more NG particles. We expect that in all these future efforts, the underlying gauge invariance will play a central role.

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- [11] The same issue was discussed briefly already in Ref. [7], concluding that “we have *one* gapped Goldstone mode for each *pair* of broken NC generators.” This statement, however, admits different interpretations, depending on the basis of generators.
- [12] The simplest example is perhaps a free nonrelativistic particle. It can be interpreted as a type-B NGB of a spontaneously broken centrally-extended ISO(2) symmetry [29]. A nonzero (negative) chemical potential for the SO(2) generator—the operator of particle number—lifts its energy, making it a mNGB. We then find $n_{\text{mNGB}} = 1 = \frac{2-0}{2}$ in accord with Eq. (7).
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